

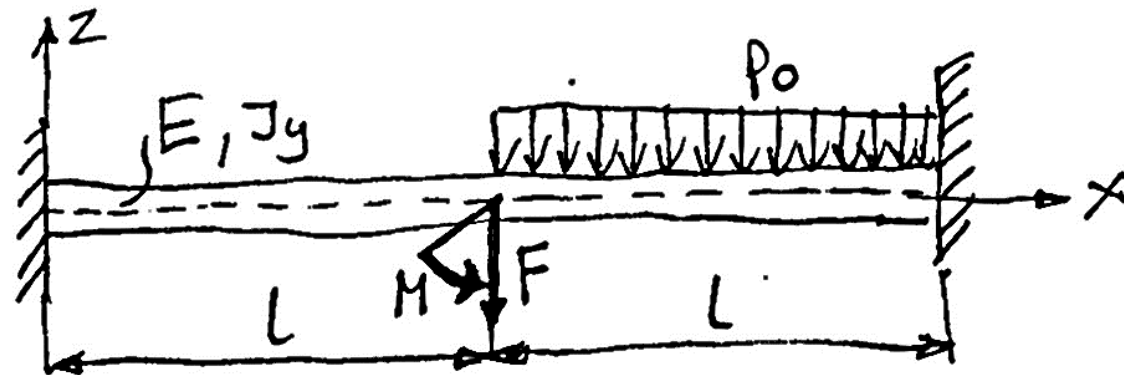


Metoda elementów skończonych (MES1)

Wykład 9B. Element belkowy - przykłady

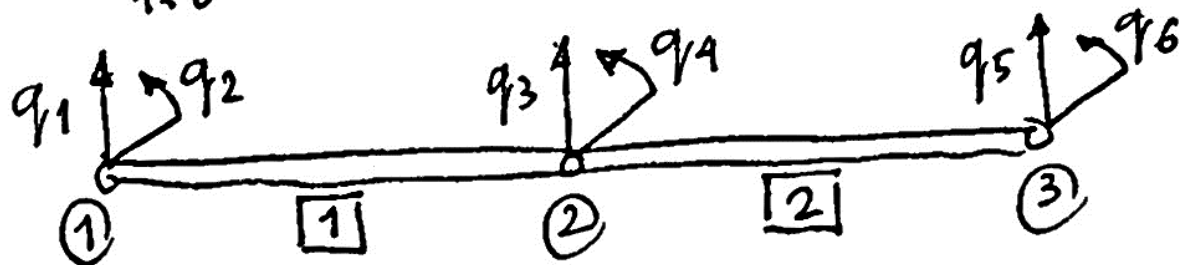
04.2022

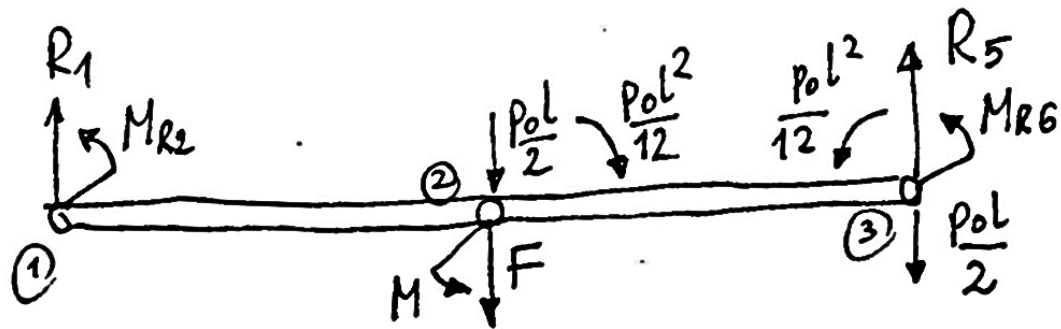
Przykład sformułuj układ równań. Znajdź przemieszczenia węzłowe, reakcje i rozkład momentu gnącego (użyj dwóch elementów)



$$L_{\underset{1 \times 6}{q}} = L [q_1, q_2, q_3, q_4, q_5, q_6]$$

Parametry
węzłowe





Siły węzłowe:

$$\mathbf{[F]}_{1 \times 6}^n = \mathbf{[R_1, M_{R2}, -F, M, R_5, M_{R6}]}$$

Siły zastępcze:

$$\begin{aligned} \mathbf{[F]}_{1 \times 6}^e &= \mathbf{[F]}_{1 \times 6}^{*1} + \mathbf{[F]}_{1 \times 6}^{*2} = \mathbf{[F_{11}, F_{21}, F_{31}, F_{41}, 0, 0]} + \\ &+ \mathbf{[0, 0, F_{12}, F_{22}, F_{32}, F_{42}]} = \\ &= \mathbf{[0+0, 0+0, 0-\frac{Pol}{2}, 0-\frac{Pol^2}{12}, 0-\frac{Pol}{2}, 0+\frac{Pol^2}{12}]} \end{aligned}$$

Globalny wektor obciążenia:

$$\begin{aligned} \mathbf{[F]}_{1 \times 6} &= \mathbf{[F]}_{1 \times 6}^n + \mathbf{[F]}_{1 \times 6}^e = \\ &= \mathbf{[R_1, M_{R2}, -F-\frac{Pol}{2}, M-\frac{Pol^2}{12}, R_5-\frac{Pol}{2}, M_{R6}+\frac{Pol^2}{12}]} = \\ &= \mathbf{[F_1, F_2, F_3, F_4, F_5, F_6]} \\ &\quad \text{(N) (Nm) ...} \end{aligned}$$

Macierze sztywności:

$$[k]_1 = [k]_2 = \frac{2EJ_y}{l^3} \begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}$$

$$[k]_1^* = \begin{bmatrix} \text{hatched} & 0 & 0 \\ \text{hatched} & [k]_1 & 0 \\ \text{hatched} & 0 & 0 \\ \text{hatched} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[k]_2^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \text{hatched} & \text{hatched} & \text{hatched} \\ 0 & 0 & \text{hatched} & \text{hatched} & \text{hatched} \\ 0 & 0 & \text{hatched} & \text{hatched} & \text{hatched} \end{bmatrix}$$

Globalna macierz sztywności:

$$[K] = [k]_1^* + [k]_2^* = \begin{bmatrix} \text{hatched} & 0 & 0 \\ \text{hatched} & [k]_1 & 0 \\ \text{hatched} & 0 & \text{hatched} \\ \text{hatched} & 0 & \text{hatched} \\ 0 & 0 & \text{hatched} \\ 0 & 0 & \text{hatched} \end{bmatrix}$$

Układ równań + warunki brzegowe:

$$[K] \cdot \{q\} = \{F\} + \text{B.C.}: q_1=0, q_2=0, q_5=0, q_6=0$$

Rozwiązanie:

$$\begin{bmatrix} \times & \times \\ \times & \times \\ \times & \times \\ \times & \times \end{bmatrix} \cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

$$\frac{2EJ_y}{l^3} \begin{bmatrix} 6+6 & -3l+3l \\ -3l+3l & 2l^2+2l^2 \end{bmatrix} \cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

$$\frac{EJ_y}{l^3} \begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix} \cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

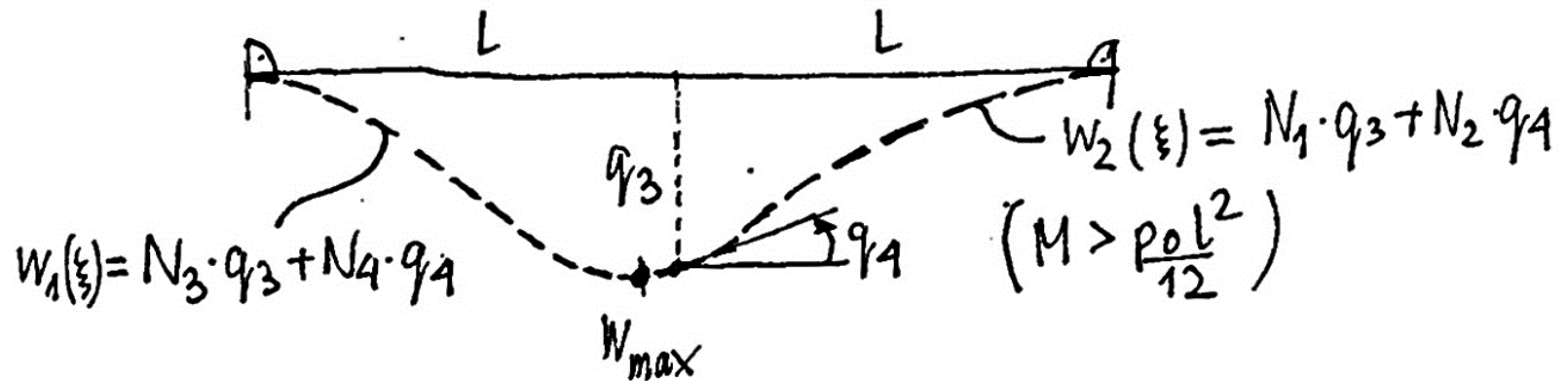
$$\frac{24EJ_y}{l^3} \cdot q_3 = F_3 \Rightarrow q_3 = -\frac{(F + \frac{P_0 l}{2}) l^3}{24EJ_y} \quad (\text{mm})$$

$$\frac{8EJ_y}{l} q_4 = F_4 \Rightarrow q_4 = \frac{(M - \frac{P_0 l^2}{12}) l}{8EJ_y} \quad (\text{rad})$$

Ugięcie belki:

$$W_1(\xi) = [N] \cdot \{q\}_1 = [N_1, N_2, N_3, N_4] \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \end{Bmatrix}_1$$

$$W_2(\xi) = [N] \cdot \{q\}_2 = [N_1, N_2, N_3, N_4] \cdot \begin{Bmatrix} q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix}_2$$



Rrakcje:

$$\frac{2EJ_y}{l^3} [6, 3l, -6, 3l, 0, 0] \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix} = F_1 = R_1$$

$$R_1 = -\frac{12EJ_y}{l^3} \cdot \left(-\frac{(F + \frac{P_0 l}{2}) l^3}{24EJ_y} \right) + \frac{6EJ_y}{l^2} \cdot \frac{(M - \frac{P_0 l}{12}) l}{8EJ_y} =$$

$$= \frac{1}{2} (F + \frac{P_0 l}{2}) + \frac{3}{4l} (M - \frac{P_0 l^2}{12}) =$$

$$= \frac{1}{2} F + \frac{3}{4l} M + \frac{3}{16} P_0 l$$

$$\frac{2EJ_y}{l^3} [3l, 2l^2, -3l, 0, 0] \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix} = F_2 = M_{R2}$$

$$M_{R2} = -\frac{6EJ_y}{l^2} \cdot \left(-\frac{(F + \frac{\rho_0 l}{2}) l^3}{24EJ_y} \right) + \frac{2EJ_y}{l} \cdot \frac{(M - \frac{\rho_0 l^2}{12}) l}{8EJ_y} =$$

$$= +\frac{1}{4} (F + \frac{\rho_0 l}{2}) l + \frac{1}{4} (M - \frac{\rho_0 l^2}{12}) = \underbrace{\frac{1}{4} Fl + \frac{1}{4} M + \frac{5}{48} \rho_0 l^2}$$

$$\frac{2EJ_y}{L^3} \begin{bmatrix} 0 & 0 & -6 & -3L & 6 & -3L \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix} = F_5 = R_5 - \frac{P_0 L}{2}$$

$$R_5 = \frac{-12EJ_y}{L^3} \cdot \left(-\frac{(F + \frac{P_0 L}{2})L^3}{24EJ_y} \right) - \frac{6EJ_y}{L^2} \frac{(M - \frac{P_0 L^2}{12})L}{8EJ_y} + \frac{P_0 L}{2} =$$

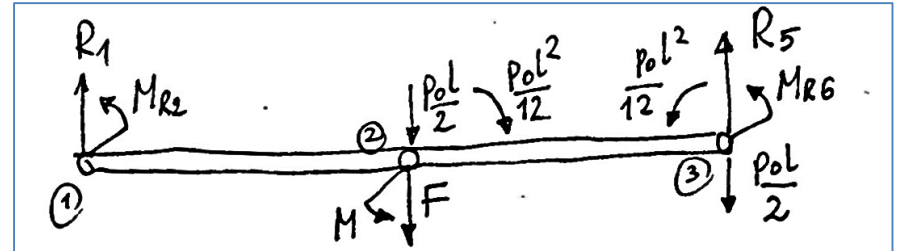
$$= \frac{1}{2}(F + \frac{P_0 L}{2}) - \frac{3}{4L}(M - \frac{P_0 L^2}{12}) + \frac{P_0 L}{2} = \frac{1}{2}F - \frac{3}{4L}M + \frac{13}{16}P_0 L$$

$$\frac{2EJ_y}{L^3} [0, 0, 3L, L^2, -3L, 2L^2] \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix} = F_6 = M_{R6} + \frac{P_0 L^2}{12}$$

$$M_{R6} = \frac{6EJ_y}{L^2} \left(-\frac{(F + \frac{P_0 L}{2}) L^3}{24EJ_y} \right) + \frac{2EJ_y}{L} \cdot \frac{(M - \frac{P_0 L^2}{12}) L}{8EJ_y} - \frac{P_0 L^2}{12} =$$

$$= -\frac{1}{4} (F + \frac{P_0 L}{2}) L + \frac{1}{4} (M - \frac{P_0 L^2}{12}) - \frac{P_0 L^2}{12} = \underbrace{-\frac{1}{4} FL + \frac{1}{4} M - \frac{11}{48} P_0 L^2}_{\text{Result}}$$

Sprawdzenie równowagi:



$$\sum F_z = 0$$

$$R_1 - \frac{P_0 L}{2} - F - \frac{P_0 L}{2} + R_5 = 0$$

$$\frac{1}{2} F + \frac{3}{4L} \cdot M + \frac{3}{16} P_0 L - \frac{P_0 L}{2} - F - \frac{P_0 L}{2} + \frac{1}{2} F - \frac{3}{4L} \cdot M + \frac{13}{16} P_0 L = 0$$

$$\sum M_y^{x=0} = 0 \quad \uparrow$$

$$M_{R2} - \frac{P_0 L}{2} \cdot L + M - F \cdot L - \frac{P_0 L^2}{12} + \frac{P_0 L^2}{12} + R_5 \cdot 2L + M_{R6} - \frac{P_0 L}{2} \cdot 2L = 0$$

$$M_{R2} - \frac{3}{2} P_0 L^2 + M - F \cdot L + R_5 \cdot 2L + M_{R6} =$$

$$= \frac{1}{4} F L + \frac{1}{4} M + \frac{5}{48} P_0 L^2 - \frac{3}{2} P_0 L^2 + M - F \cdot L + F \cdot L - \frac{3}{2} M + \frac{13}{8} P_0 L^2 +$$

$$- \frac{1}{4} F L + \frac{1}{4} M - \frac{11}{48} P_0 L^2 = \frac{5}{48} P_0 L^2 - \frac{3}{2} P_0 L^2 + \frac{13}{8} P_0 L^2 - \frac{11}{48} P_0 L^2$$

$$= \frac{5 - 3 \cdot 24 + 13 \cdot 6 - 11}{48} P_0 L^2 = \frac{5 - 72 + 78 - 11}{48} P_0 L^2 = 0$$

Moment gąący:

BENDING MOMENT:

$$\boxed{1}: M_{y_1}(\xi) = EJ_y W_1'' = EJ_y \underline{N''} \cdot \{q\}_1 =$$

$$= EJ_y \underline{[N_1'', N_2'', N_3'', N_4'']} \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \end{Bmatrix} = EJ_y (N_3'' \cdot q_3 + N_4'' \cdot q_4) =$$

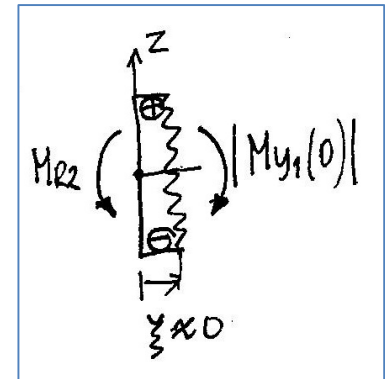
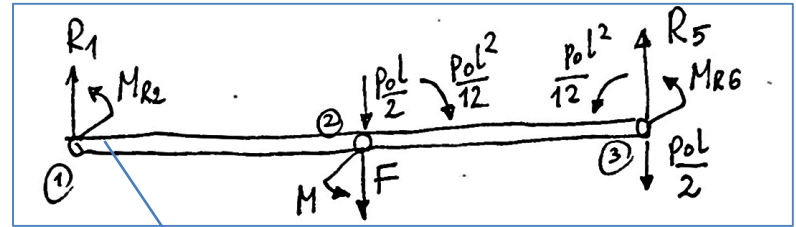
$$= EJ_y \left(\left(\frac{6}{l^2} - \frac{12}{l^3} \xi \right) q_3 + \left(-\frac{2}{l} + \frac{6}{l^2} \cdot \xi \right) \cdot q_4 \right)$$

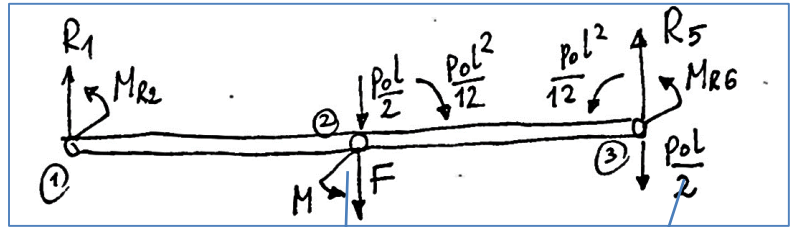
$$M_{y_1}(0) = EJ_y \left(\frac{6q_3}{l^2} - \frac{2q_4}{l} \right) = -\frac{6(F + \frac{\rho_0 l}{2})l^3}{24l^2} - \frac{2(M - \frac{\rho_0 l^2}{12})l}{8l} =$$

$$= -\frac{1}{4} \left(Fl + \frac{5}{12} \rho_0 l^2 + M \right) \quad (= -M_{R2})$$

$$M_{y_1}(l) = EJ_y \left(\left(\frac{6}{l^2} - \frac{12}{l^2} \right) q_3 + \left(-\frac{2}{l} + \frac{6}{l} \right) q_4 \right) = EJ_y \left(-\frac{6}{l^2} \cdot q_3 + \frac{4}{l} q_4 \right) =$$

$$= \frac{6(F + \frac{\rho_0 l}{2})l^3}{24l^2} + \frac{4(M - \frac{\rho_0 l^2}{12})l}{8l} = \frac{1}{4} \left(F \cdot l + \frac{1}{3} \rho_0 l^2 + 2M \right)$$





$$\square 2 : M_{y_2}\left(\frac{l}{3}\right) = EJ_y w_2'' = EJ_y [N_1'' N_2'' N_3'' N_4''] \cdot \begin{Bmatrix} q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix}_2 =$$

$$= EJ_y (N_1'' \cdot q_3 + N_2'' \cdot q_4) = EJ_y \left(\left(-\frac{6}{l^2} + \frac{12}{l^3} \frac{l}{3} \right) q_3 + \left(-\frac{4}{l} + \frac{6}{l^2} \frac{l}{3} \right) q_4 \right)$$

$$M_{y_2}(0) = EJ_y \left(-\frac{6}{l^2} q_3 - \frac{4}{l} q_4 \right) = \frac{6(F + \frac{p_0 l}{2}) l^3}{24 l^2} - \frac{4(M - \frac{p_0 l^2}{12}) l}{8 l} =$$

$$= \frac{1}{4} (F l + \frac{2}{3} p_0 l^2 - 2M)$$

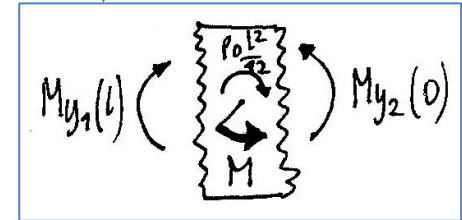
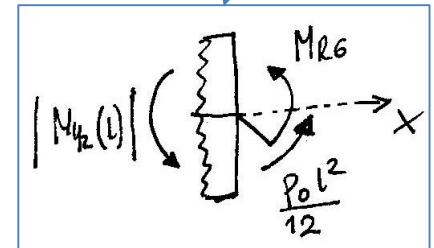
$$M_{y_2}(l) = EJ_y \left(\frac{6}{l^2} q_3 + \frac{2}{l} q_4 \right) = -\frac{6(F + \frac{p_0 l}{2}) l^3}{24 l^2} + \frac{2(M - \frac{p_0 l^2}{12}) l}{8 l} =$$

$$= -\frac{1}{4} (F \cdot l + \frac{2}{3} p_0 l^2 - M)$$

$$(= MR_6 + \frac{p_0 l^2}{12})$$

$$M_{y_1}(l) - M_{y_2}(0) = \frac{1}{4} (F \cdot l + \frac{2}{3} p_0 l^2 + 2M) - \frac{1}{4} (F \cdot l + \frac{2}{3} p_0 l^2 - 2M) =$$

$$= M - \frac{p_0 l^2}{12}$$



Założmy:

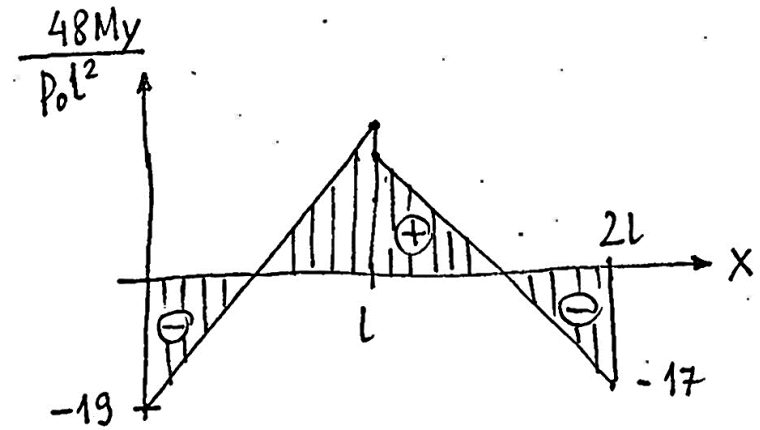
$$F = p_0 \cdot L, \quad M = \frac{p_0 l^2}{6}$$

$$M_{y_1}(0) = -\frac{1}{4} \left(p_0 l^2 + \frac{5}{12} p_0 l^2 + \frac{p_0 l^2}{6} \right) = -\frac{19}{48} p_0 l^2$$

$$M_{y_1}(L) = \frac{1}{4} \left(p_0 l^2 + \frac{1}{3} p_0 l^2 + \frac{1}{3} p_0 l^2 \right) = \frac{20}{48} p_0 l^2$$

$$M_{y_2}(0) = \frac{1}{4} \left(p_0 l^2 + \frac{2}{3} p_0 l^2 - \frac{1}{3} p_0 l^2 \right) = \frac{16}{48} p_0 l^2$$

$$M_{y_2}(L) = -\frac{1}{4} \left(p_0 l^2 + \frac{7}{12} p_0 l^2 - \frac{p_0 l^2}{6} \right) = -\frac{17}{48} p_0 l^2$$



Naprężenia normalne:

